Correspondence Between Unicritical and Maximally Critical Laminations of Degree d

Brittany Burdette

Department of Mathematics
University of Alabama at Birmingham

May 17, 2022

Topology and Dynamical Systems Workshop

Outline

1. Background Information
2. The Correspondence
3. Future and Current Work

Outline

1. Background Information
2. The Correspondence
3. Future and Current Work

Outline

Background Definitions

Definition (Lamination)

A *lamination* of the closed unit disk, $\overline{\mathbb{D}}$, is a closed collection of chords with the property that no two chords intersect within the (open) unit disk.

Definition (Sibling Invariant Lamination)

A lamination $\mathcal{L}$ is said to be *sibling d-invariant* (or simply *invariant* if no confusion will result) provided that

1. (Forward Invariant) For every $\ell \in \mathcal{L}$, $\sigma_d(\ell) \in \mathcal{L}$.
2. (Backward Invariant) For every non-degenerate $\ell' \in \mathcal{L}$, there is a leaf $\ell \in \mathcal{L}$ such that $\sigma_d(\ell) = \ell'$.
3. (Sibling Invariant) For every $\ell_1 \in \mathcal{L}$ with $\sigma_d(\ell_1) = \ell'$, a non-degenerate leaf, there is a full sibling collection $\{\ell_1, \ell_2, \dots, \ell_d\} \subset \mathcal{L}$ such that $\sigma_d(\ell_i) = \ell'$.

Background Definitions

Definition (Gap)

A *gap* of a lamination is the closure of a component of $\overline{\mathbb{D}} \setminus \cup\mathcal{L}$.

- Finite Gaps are called polygons

Definition (Fatou Gap)

A gap whose intersection with the circle contains a Cantor set is called a Fatou gap.

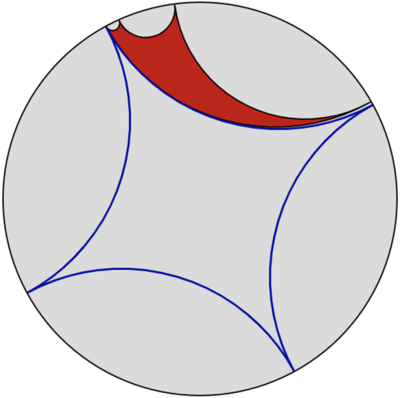
Definition (Co-root)

A *co-root* is a point, other than an endpoint of the major, in the boundary of the central gap of the unicritical lamination that is fixed under the first return map. Alternatively, it is also referred to as the image of these points (under the minor instead of the major).

Background Definitions

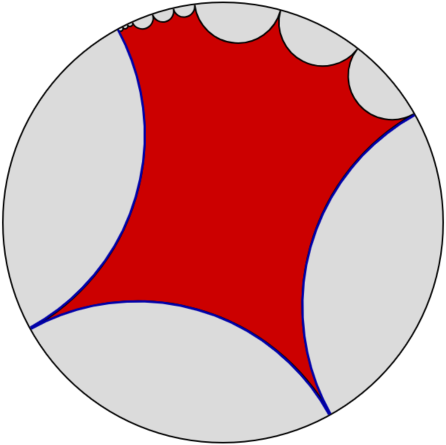
Definition (Unicritical)

Unicritical Polygons have all criticality on exactly one side of the polygon which results in a degree *d* Fatou gap.



Definition (Maximally Critical)

Maximally Critical Polygons have $d-1$ degree 2 Fatou gaps.

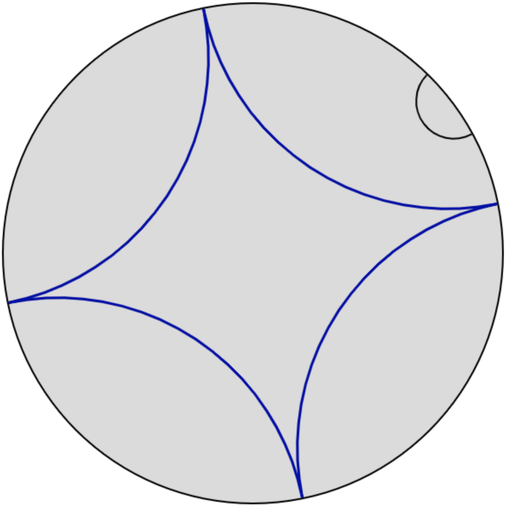


The Correspondence

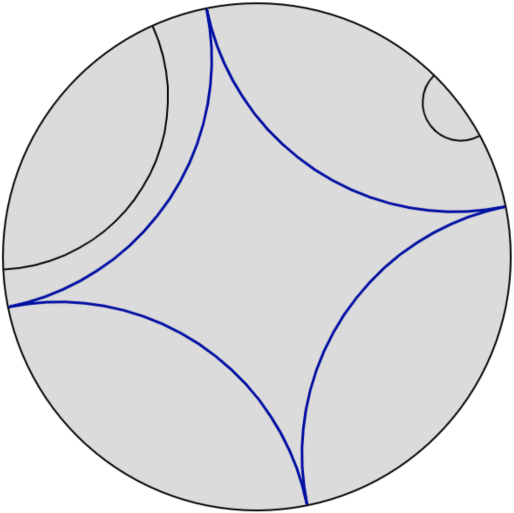
Theorem

There is a 1-1 correspondence between Unicritical Polygons and Maximally Critical Polygons. In the identity return case, the maximally critical polygon will have d sides. In the rotational and rotation return case, the maximally critical polygon will have adjacent majors and will have $k(d-1)$ sides where k is the period of the unicritical major leaf.

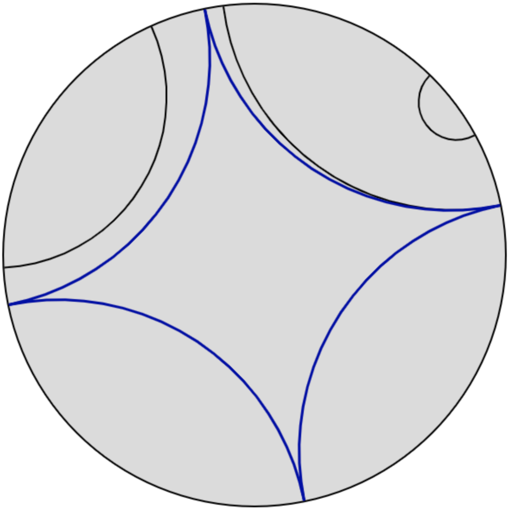
Identity Return Case: Unicritical to Maximally Critical



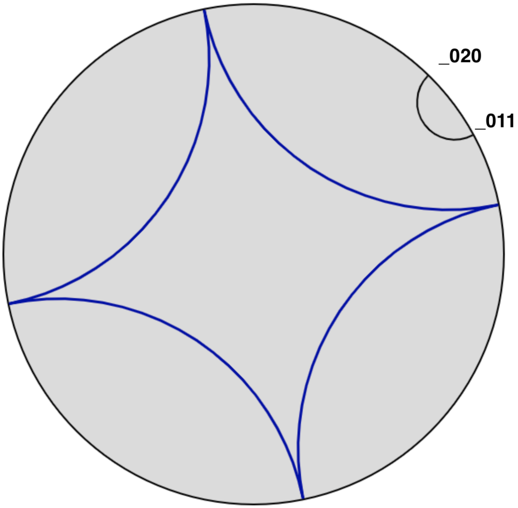
Identity Return Case: Unicritical to Maximally Critical



Identity Return Case: Unicritical to Maximally Critical

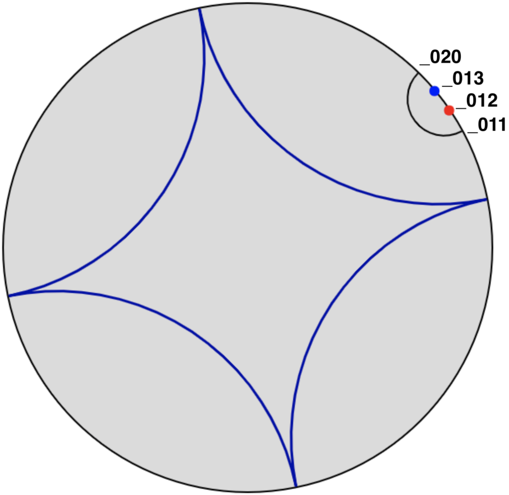


Identity Return Case: Unicritical to Maximally Critical



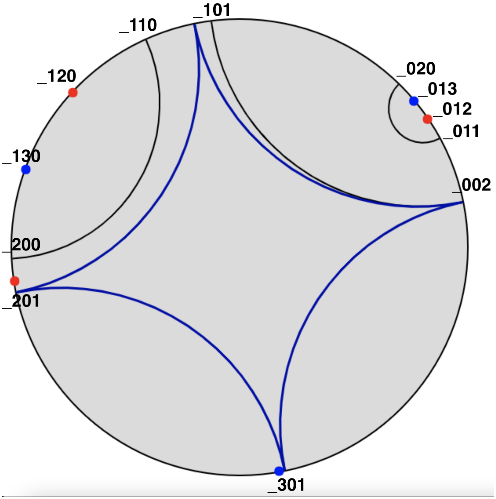
Identity Return Case: Unicritical to Maximally Critical

Using the Generalized Lavaur's Algorithm, we can identity the co-roots: _012 and _013



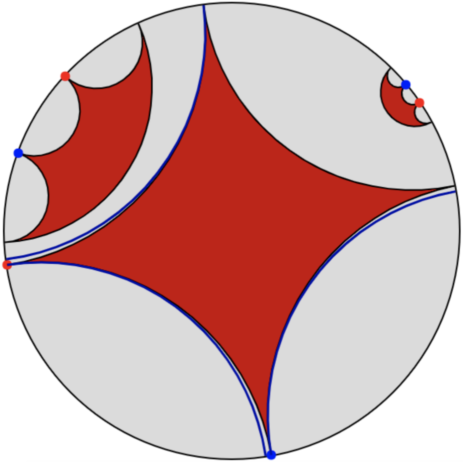
Identity Return Case: Unicritical to Maximally Critical

Now, we add the full forward orbit of the identity return leaf and co-roots.



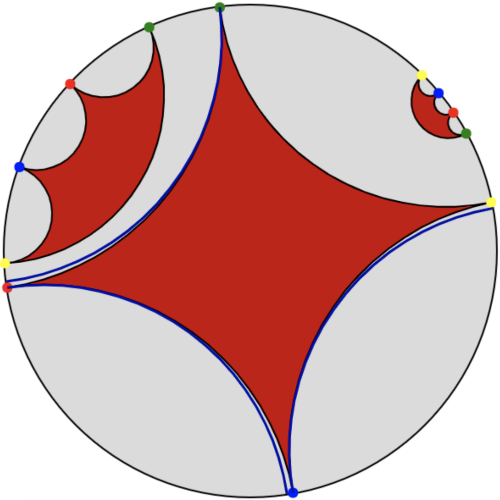
Identity Return Case: Unicritical to Maximally Critical

When we connect the ends of our Identity Return leaf with the corresponding co-roots, we get a maximally critical d-gon.



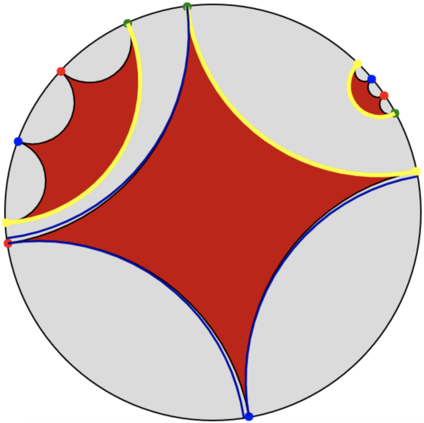
Identity Return Case: Maximally Critical to Unicritical

A maximally critical polygon and its two forward images:

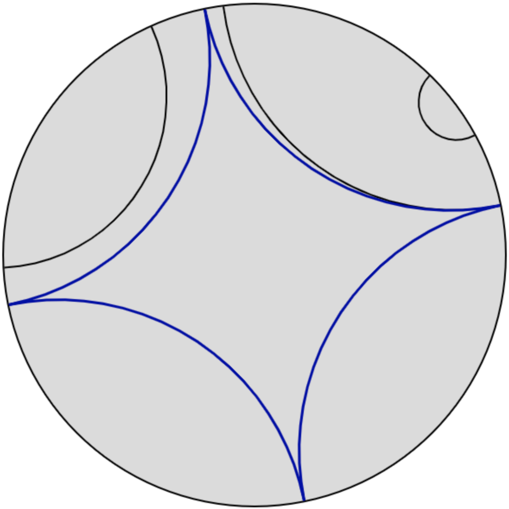


Identity Return Case: Maximally Critical to Unicritical

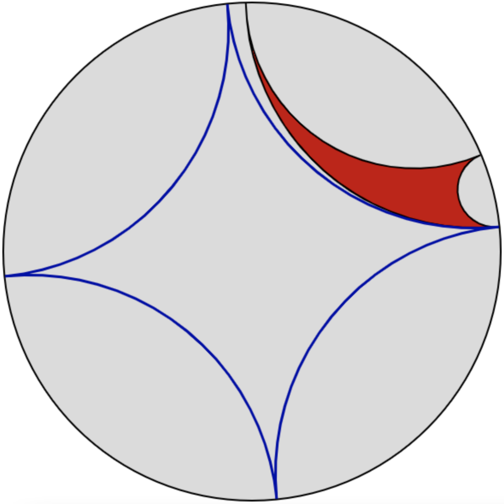
In order to recover our original major, we need to identify which leaf is shorter than $\frac{1}{d}$. This leaf will be the unicritical major and is still visible in the maximally critical case.



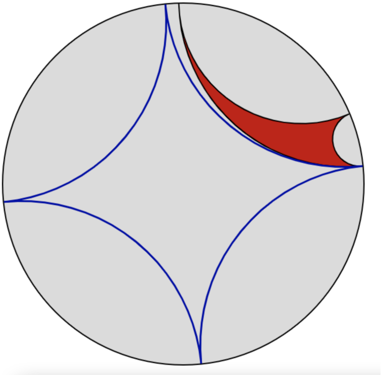
Identity Return Case: Maximally Critical to Unicritical

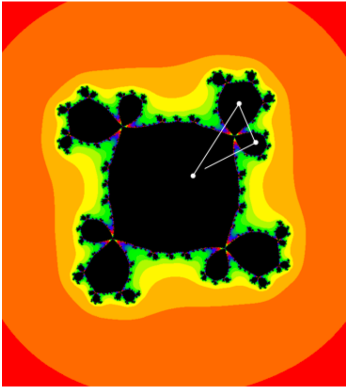


Rotational Case: Unicritical to Maximally Critical

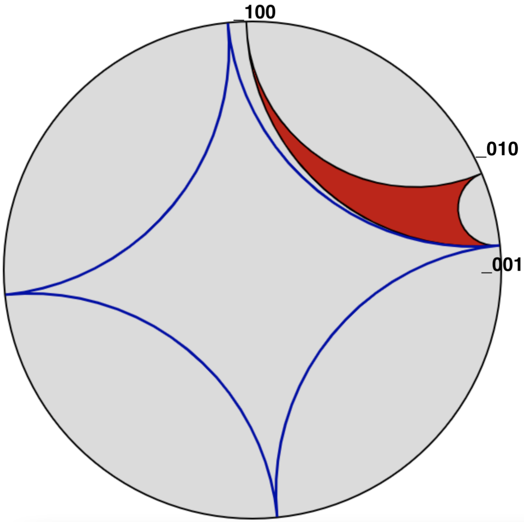


Julia Set for Unicritical Rotation Polygon



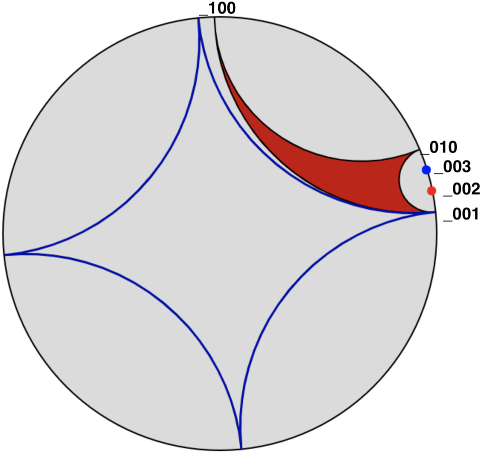


Rotational Case: Unicritical to Maximally Critical



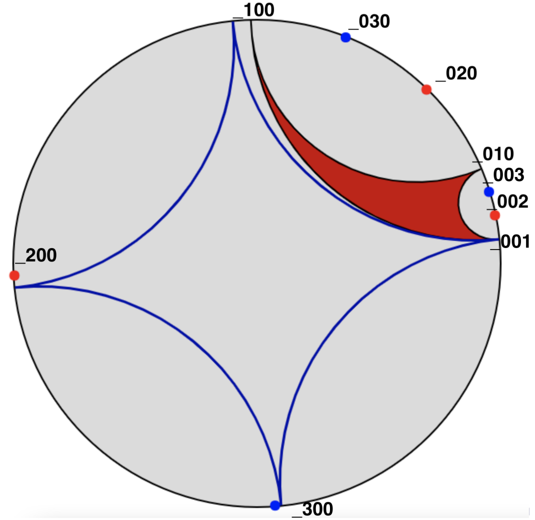
Rotational Case: Unicritical to Maximally Critical

Using the Generalized Lavaur's Algorithm for determining co-roots of minors of unicritical laminations, we have that the co-roots are _002 and _003.



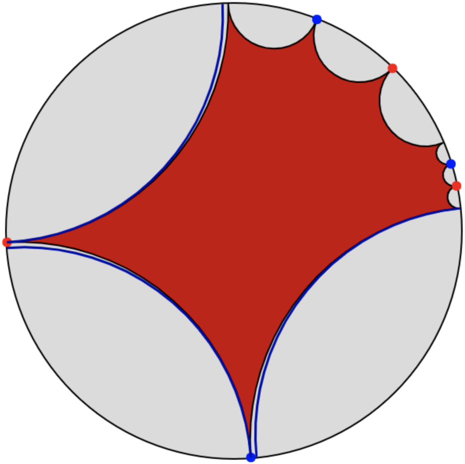
Rotational Case: Unicritical to Maximally Critical

Now, we add the full forward orbit of the co-roots.



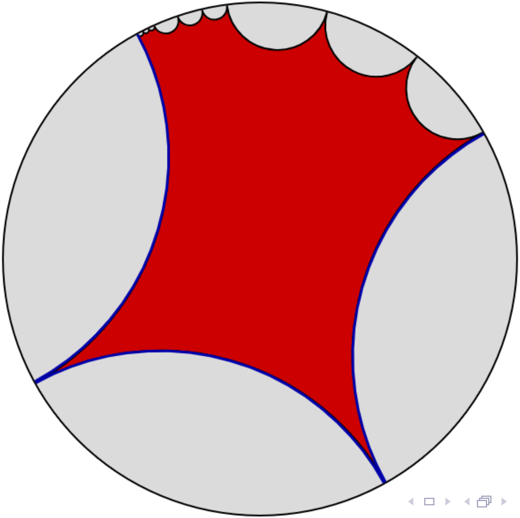
Rotational Case: Unicritical to Maximally Critical

By connecting the co-roots, their forward images, and our original vertices, we find our maximally critical polygon.



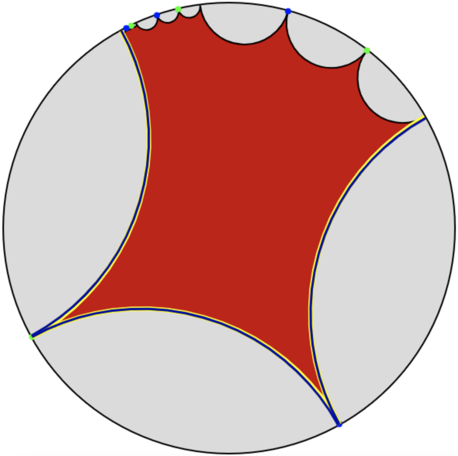
Rotational Case: Maximally Critical to Unicritical

We start with a maximally critical rotational polygon.



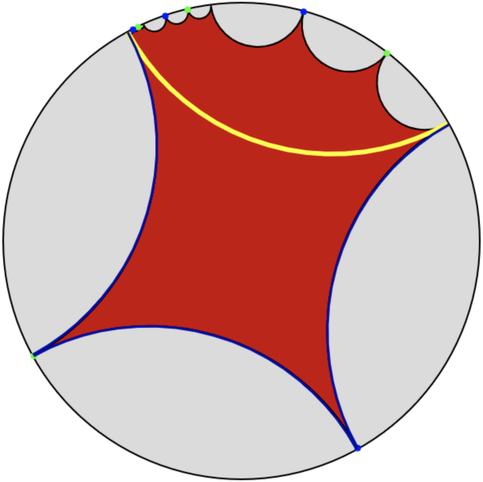
Rotational Case: Maximally Critical to Unicritical

In this example, we have 3 orbits of vertices and chain of adjacent major leaves shown in yellow.

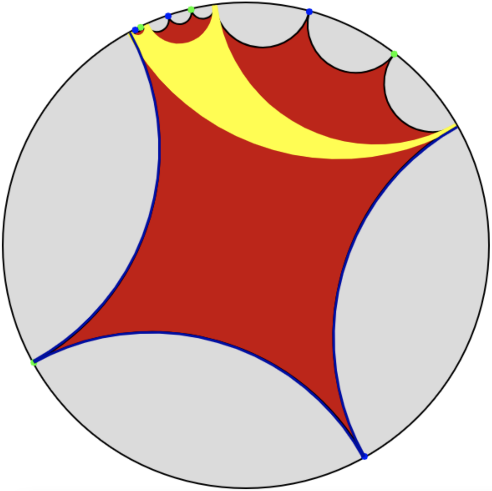


Rotational Case: Maximally Critical to Unicritical

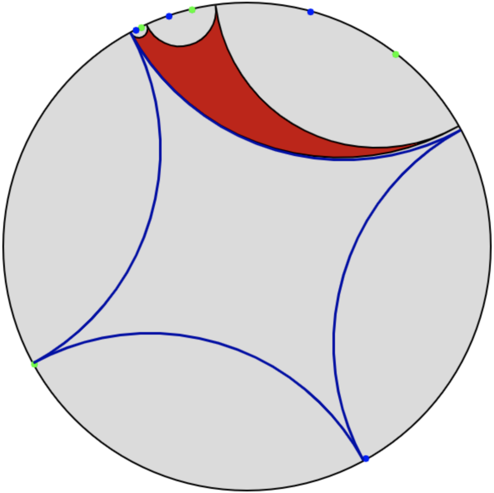
We connect the beginning and end of the chain of major leaves to recover our unicritical major leaf.



Rotational Case: Maximally Critical to Unicritical

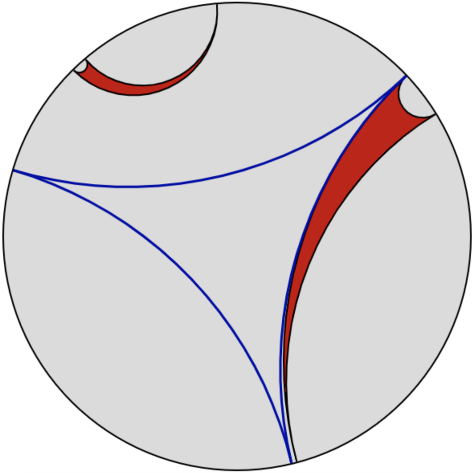


Rotational Case: Maximally Critical to Unicritical



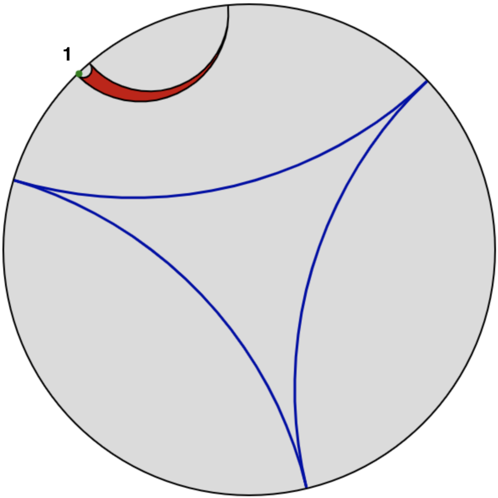
Rotation Return Case: Unicritical to Maximally Critical

This is a unicritical rotation return polygon. Note that only one leaf experiences criticality.

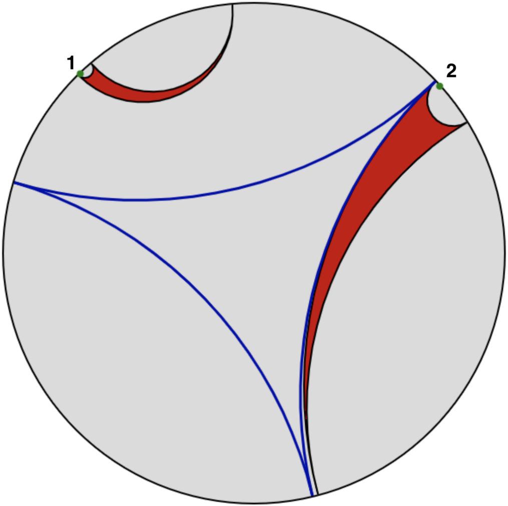


Rotation Return Case: Unicritical to Maximally Critical

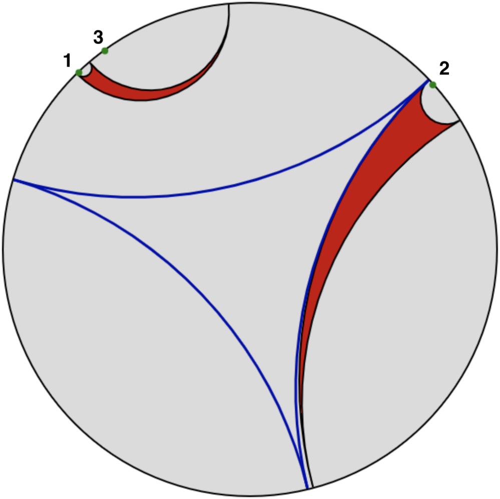
Again, using the Generalized Lavaur's Algorithm, we find the co-root under the minor leaf.



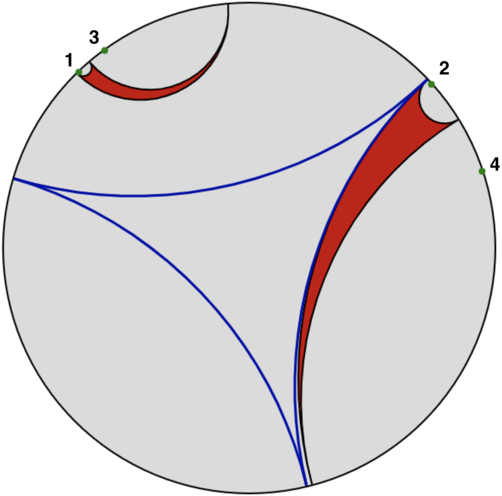
Rotation Return Case: Unicritical to Maximally Critical



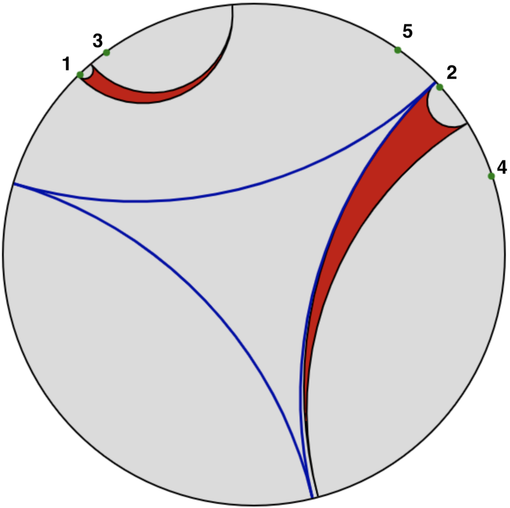
Rotation Return Case: Unicritical to Maximally Critical



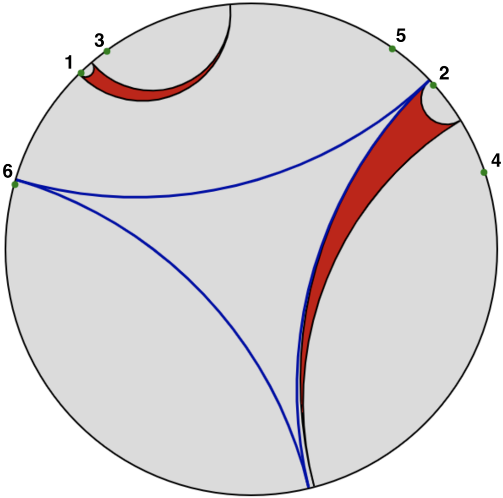
Rotation Return Case: Unicritical to Maximally Critical



Rotation Return Case: Unicritical to Maximally Critical

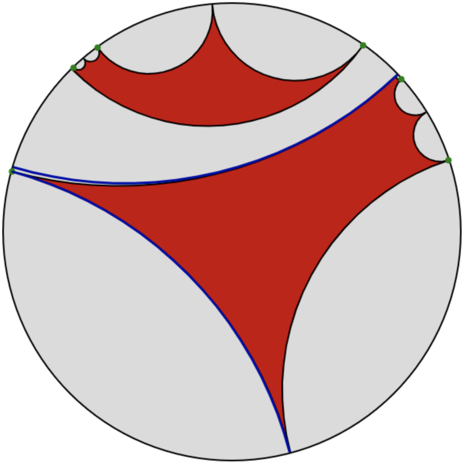


Rotation Return Case: Unicritical to Maximally Critical



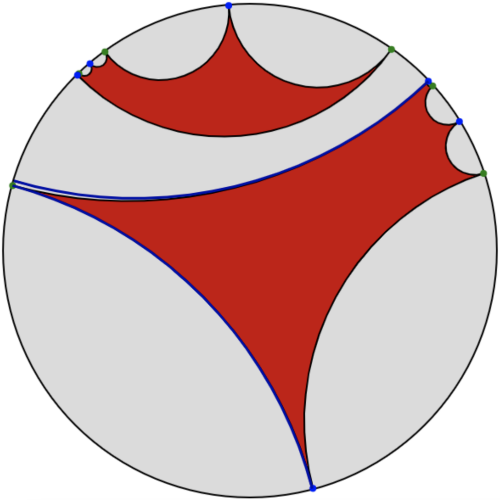
Rotation Return Case: Unicritical to Maximally Critical

We now join the endpoints of each polygon with the corresponding co-root images.



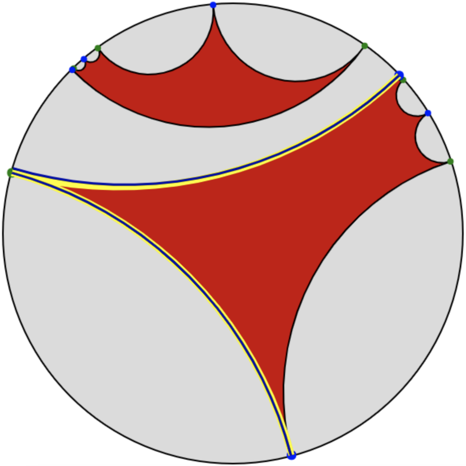
Rotation Return Case: Maximally Critical to Unicritical

Now, we start with a maximally critical rotation return polygon.

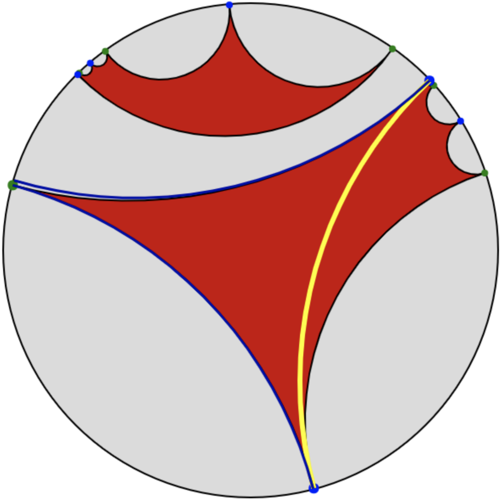


Rotation Return Case: Maximally Critical to Unicritical

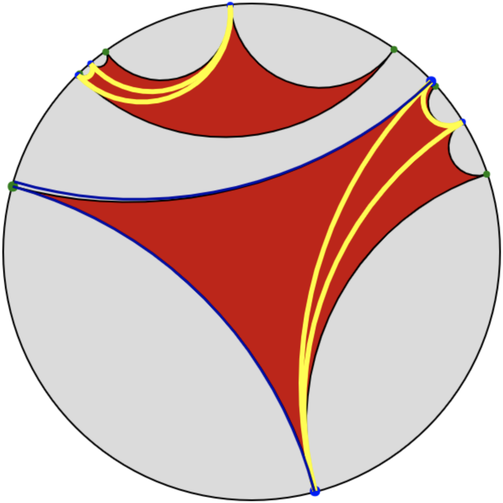
Similar to the rotation case, we identify the chain of adjacent major leaves shown in yellow:



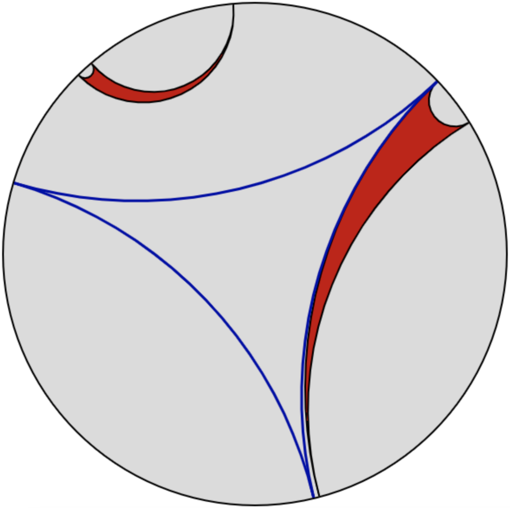
Rotation Return Case: Maximally Critical to Unicritical



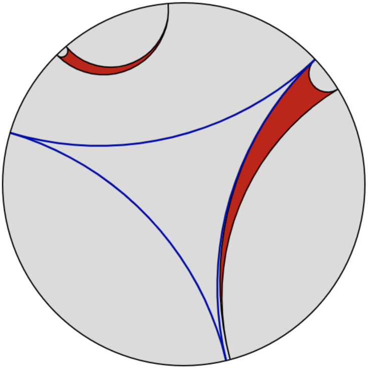
Rotation Return Case: Maximally Critical to Unicritical

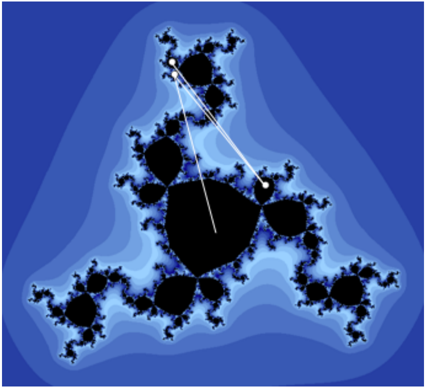


Rotation Return Case: Maximally Critical to Unicritical



Julia Set for Our Unicritical Rotation Return Polygon





Future and Current Work

Current Work (all Single Critical Moment):

- Identity Return (Cameron Hale)
- Rotational with Adjacent Majors
- Rotation Return with Adjacent Majors

Future Work:

- Rotational Single Critical Moment with Non-Adjacent Majors
- Rotational Multi-Critical Moment
- Rotation Return Single Critical Moment with Non-Adjacent Majors
- Rotation Return Multi-Critical Moment

References

- B. Barry. On the simplest lamination of a given identity return triangle. MS thesis UAB 2015.
- S. Bhattacharya. Topics in Low Dimensional Dynamical Systems : Interval Rotation Numbers and Laminations. PhD thesis UAB 2021.
- C. Hale. Unicritical Laminations and d-gons of Single Critical Moment. MS thesis UAB 2020.
- A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg. Laminations in the Language of Leaves. *Trans. Amer. Math. Soc.* 365 (2013), 5367–5391.
- W. Thurston. On the geometry and dynamics of iterated maps. in *Complex Dynamics; Families and Friends* edited by D. Schliecher (A K Peters, 2009).